

SMALL DOUBLING IN ORDERED NILPOTENT GROUPS OF CLASS 2

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ABSTRACT. The aim of this paper is to present a complete description of the structure of finite subsets S of a nilpotent group of class 2 satisfying $|S^2| = 3|S| - 2$.

1. Introduction.

Let α and β denote real numbers, with $\alpha > 1$. A finite subset S of a group G is said to satisfy the *small doubling property* if

$$|S^2| \leq \alpha|S| + \beta,$$

where $S^2 = \{s_1s_2 \mid s_1, s_2 \in S\}$.

The classical Freiman's inverse theorems describe the structure of finite subsets of abelian groups, which satisfy the small doubling property (see [3] [4], [5], [6] [15] and [18]). Recently, several authors obtained similar results concerning various classes of groups for an arbitrary α (see for example [1], [2], [7], [12], [13], [17], [19] and [20]).

In [7] we started the investigation of finite subsets of *ordered groups* satisfying the small doubling property with $\alpha = 3$ and small $|\beta|$'s. We proved that if $(G, <)$ is an ordered group and S is a finite subset of G of size $k \geq 2$, such that $|S^2| \leq 3k - 3$, then $\langle S \rangle$ is abelian. Furthermore, if $k \geq 3$ and $|S^2| \leq 3k - 4$, then there exist $x_1, g \in G$ such that $g > 1$, $gx_1 = x_1g$ and S is a subset of the geometric progression $\{x_1, x_1g, x_1g^2, \dots, x_1g^{t-k}\}$, where $t = |S^2|$. We also showed that these results are best possible, by presenting an example of an ordered group with a subset S of size k with $\langle S \rangle$ non-abelian and $|S^2| = 3k - 2$.

Other recent results concerning small doubling properties appear in [8], [9], [10].

In this paper we study the structure of finite subsets with small doubling in torsion-free nilpotent groups of class 2. It is known that

2000 *Mathematics Subject Classification*. Primary: 20F60, 20F18; Secondary: 20F99, Primary 11P70; Secondary 11B75, 52C99, 05D99.

Key words and phrases. ordered groups, finite subsets, small doubling, nilpotent groups.

these groups are orderable (see [14] and [16]), so the previous results apply.

Our main aim is to completely describe the structure of subsets S of size k in ordered nilpotent groups of class 2, satisfying $|S^2| = 3k - 2$. In particular, we show in Theorem 3.2 that if $\langle S \rangle$ is non-abelian and $|S| = k \geq 4$, then $|S^2| = 3k - 2$ if and only if $S = \{a, ac, ac^2, \dots, ac^i, b, bc, bc^2, \dots, bc^j\}$, where $a, b, c \in G$, i, j are non-negative integers, $c > 1$ and either $ab = bac$ or $ba = abc$.

In Theorem 2.5 we describe the structure of subsets S of size $k \geq 4$ in such groups, with $|S^2| = 3k + i \leq 4k - 6$.

In a forthcoming paper we shall describe subsets S of size k in an ordered nilpotent group of class 2 satisfying $|S^2| = 3k - 1$.

2. Some general results.

We start with the following very useful lemma.

Lemma 2.1 *Let $(G, <)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 3$, satisfying:*

$$S = \{x_1, \dots, x_k\}, \quad x_1 < x_2 < \dots < x_k$$

and

$$x_k x_{k-1} \neq x_{k-1} x_k.$$

Let $T = \{x_1, \dots, x_{k-1}\}$. Then:

$$|S^2| \geq |T^2| + 4.$$

In particular, if $\langle T \rangle$ is non-abelian, then

$$|S^2| \geq 3k - 1.$$

Proof. Write $T = \{x_1, x_2, \dots, x_{k-1}\}$ and let

$$D = \{x_k^2, x_k x_{k-1}, x_{k-1} x_k\}.$$

Then $|D| = 3$ and $D \subseteq S^2 \setminus T^2$, since $x_i x_j \leq x_{k-1} x_{k-1}$ for each $x_i x_j \in T^2$. If either $x_k x_{k-2}$ or $x_{k-2} x_k$ does not belong to $T^2 \dot{\cup} D$, then

$$|S^2 \setminus T^2| \geq 4,$$

as required. So we may assume, from now on, that

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subset T^2 \dot{\cup} D.$$

Our aim is to reach a contradiction.

First, we claim that the case

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subseteq T^2$$

is impossible. Indeed, in such case

$$x_k x_{k-2} = x_j x_{k-1} \text{ and } x_{k-2} x_k = x_{k-1} x_i \text{ for some } 1 \leq i, j \leq k-1.$$

Suppose, first, that $i, j \leq k-2$. Then $x_{k-2} x_k = x_k x_{k-2} [x_{k-2}, x_k] = x_{k-1} x_i$ with $k > k-1$ and $k-2 \geq i$. Hence $[x_{k-2}, x_k] < 1$ and $[x_k, x_{k-2}] > 1$, yielding

$$x_j x_{k-1} = x_k x_{k-2} = x_{k-2} x_k [x_k, x_{k-2}] > x_{k-2} x_k$$

with $x_j \leq x_{k-2}$ and $x_{k-1} < x_k$, a contradiction. Thus either $i = k-1$ or $j = k-1$.

Assume that $j = k-1$. Then $x_k x_{k-2} = x_{k-1}^2$ and

$$x_{k-2} x_k = x_k x_{k-2} [x_{k-2}, x_k] = x_{k-1}^2 [x_{k-2}, x_k],$$

which implies that x_{k-1} centralizes $x_{k-2} x_k$, since G has class 2. Therefore $x_{k-2} x_k = x_{k-1} x_i = x_i x_{k-1}$, forcing $i = k-1$ and

$$x_k x_{k-2} = x_{k-1}^2 = x_{k-2} x_k.$$

Thus x_k centralizes x_{k-2} and hence x_k centralizes x_{k-1}^2 . But then x_k centralizes x_{k-1} , a contradiction. Similarly, we get a contradiction if we assume that $i = k-1$. This completes the proof of our claim.

Thus either $x_{k-2} x_k$ or $x_k x_{k-2}$ is not in T^2 and hence it belongs to D . We claim now that

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subset D.$$

Indeed, assume that $x_k x_{k-2} \in D$, which implies that

$$x_k x_{k-2} = x_{k-1} x_k.$$

Then $x_{k-2} = x_k^{-1} x_{k-1} x_k = x_{k-1} c$, where $c \in Z(G)$. If $x_{k-2} x_k \in T^2$, then $x_{k-2} x_k = x_{k-1} x_i$ for some $i \leq k-1$, which implies that $x_{k-1} c x_k = x_{k-1} x_i$ and $x_k = c^{-1} x_i$. It follows from $x_{k-2} x_k = x_{k-1} x_i$ that $x_k x_{k-2} [x_{k-2}, x_k] = x_i x_{k-1} [x_{k-1}, x_i]$. But $[x_{k-2}, x_k] = [x_{k-1} c, c^{-1} x_i] = [x_{k-1}, x_i]$, so $x_k x_{k-2} = x_i x_{k-1} \in T^2$, a contradiction. Thus $x_k x_{k-2} \in D$ implies that $x_{k-2} x_k \in D$ and similarly $x_{k-2} x_k \in D$ implies that $x_k x_{k-2} \in D$, as claimed.

Finally, we claim that

$$\{x_k x_{k-2}, x_{k-2} x_k\} \subset D.$$

is impossible. Indeed, if that is the case, then $x_k x_{k-2} = x_{k-1} x_k$ and $x_{k-2} x_k = x_k x_{k-1}$, which implies that

$$x_{k-1} = x_k^{x_{k-2}} = x_k^{x_{k-2}^{-1}}.$$

Thus $[x_{k-2}, x_k^2] = 1$ and hence $[x_{k-2}, x_k] = 1$, which implies that $x_{k-2} x_k = (x_{k-2} x_k)^{x_k} = x_{k-1} x_k$, a final contradiction.

Therefore under our assumptions $|S^2| \geq |T^2| + 4$. In particular, if $\langle T \rangle$ is non-abelian, then by Theorem 1.3 of [7] we get that $|T^2| \geq 3(k-1) - 2 = 3k - 5$ and hence $|S^2| \geq 3k - 1$. \square

We say that a subset S of a group G is completely-non-abelian ($S \in CNA$ in short) if $ab \neq ba$ for any $a, b \in S$, $a \neq b$. As an easy consequence of Lemma 2.1 we get the following result.

Proposition 2.2 *Let S be a CNA-subset of size k of an ordered nilpotent group of class 2. Then:*

$$|S^2| \geq 4k - 4.$$

Proof. The result is certainly true if $k = 1$. If $k = 2$ and $S = \{a, b\}$, then $S^2 = \{a^2, ab, ba, b^2\}$ and they are all distinct. Hence $|S^2| = 4 = 4 \cdot 2 - 4$. So let $|S| = k \geq 3$, and suppose that the result holds for $k - 1$. Let $S = \{x_1, \dots, x_k\}$, $x_1 < x_2 < \dots < x_k$ and let $T = \{x_1, \dots, x_{k-1}\}$. Then, by Lemma 2.1, $|S^2| \geq |T^2| + 4 \geq 4(k-1) - 4 + 4 = 4k - 4$, as required. \square

The following two observations will be used repeatedly.

Lemma 2.3 *Let G be an ordered nilpotent group of class 2. Let $a, b, c \in G$ and consider the subset*

$$S = \{a, ac, \dots, ac^i, b\}$$

of G for some $i \in \mathbb{N}$. Write $A = \{a, ac, \dots, ac^i\}$. If $c > 1$ and either $ab = bac^v$ or $ba = abc^v$ for some $v \in \mathbb{N}$ satisfying $v \leq i$, then $bA \cup Ab = \{ab, abc, abc^2, \dots, abc^{i+v}\}$.

In particular, $|bA \cup Ab| = i + v + 1$ and

$$|S^2| = 3|S| + (v - 3).$$

Proof. Suppose, first, that $ba = abc^v$. Then $c^v \in Z(G)$ and hence $c \in Z(G)$. Thus A is abelian and $b \notin C_G(A)$. We have

$$Ab = \{ab, abc, \dots, abc^i\} \text{ and } bA = \{ba = abc^v, \dots, abc^{v+i}\},$$

so $Ab \cup bA = \{ab, abc, \dots, abc^{v+i}\}$ since $v \leq i$.

Therefore $|Ab \cup bA| = i + v + 1$. Furthermore $|A^2| = 2(i+1) - 1 = 2i + 1$ and

$$S^2 = A^2 \cup \{b^2\} \cup (bA \cup Ab),$$

with $\emptyset = A^2 \cap \{b^2\} = A^2 \cap (bA \cup Ab) = \{b^2\} \cap (bA \cup Ab)$, since $b \notin C_G(A)$. Hence:

$$|S^2| = 2i + 1 + 1 + i + v + 1 = 3(i + 1) + v = 3(|S| - 1) + v = 3|S| + (v - 3),$$

as required. The case $ab = bac^v$ can be dealt with similarly. \square

Lemma 2.4 *Let G be an ordered nilpotent group of class 2. Let $a, b, c \in G$ and consider the subset*

$$S = \{a, ac, \dots, ac^i, b, bc, \dots, bc^j\}$$

of G for some non-negative integers i, j satisfying $i + j \geq 1$. Write $A = \{a, ac, \dots, ac^i\}$ and $B = \{b, bc, \dots, bc^j\}$.

If $c > 1$ and either $ab = bac^v$ or $ba = abc^v$ for some $v \in \mathbb{N}$ satisfying $v \leq i + j$, then

$$AB \cup BA = \{abc^l, l \in \{0, \dots, i + j + v\}\}.$$

In particular $|AB \cup BA| = i + j + v + 1$ and

$$|S^2| = 3|S| + (v - 3).$$

Proof. Suppose, first, that $ba = abc^v$. Then $c^v \in Z(G)$ and hence $c \in Z(G)$. Thus A and B are abelian and $a \notin C_G(B)$, $b \notin C_G(A)$. We have $AB = \{ab, abc, \dots, abc^{i+j}\}$ and $BA = \{ba = abc^v, \dots, bac^{v+i+j}\}$, so

$$AB \cup BA = \{ab, abc, \dots, abc^{v+i+j}\},$$

since $v \leq i + j$. Therefore $|AB \cup BA| = i + j + v + 1$. Furthermore, $|A^2| = 2(i + 1) - 1 = 2i + 1$, $|B^2| = 2j + 1$, and

$$S^2 = A^2 \cup B^2 \cup (AB \cup BA),$$

with $\emptyset = A^2 \cap B^2 = A^2 \cap (AB \cup BA) = B^2 \cap (AB \cup BA)$, since $b \notin C_G(A)$ and $a \notin C_G(B)$. Hence:

$$|S^2| = 2i + 1 + 2j + 1 + i + j + v + 1 = 3i + 3j + 3 + v = 3|S| + (v - 3),$$

as required. The case $ab = bac^v$ can be dealt with similarly. \square

While studying subsets S of size k of ordered nilpotent groups of class 2 with the small doubling property, we shall often try to reduce the hypotheses to those of the following theorem.

Theorem 2.5 *Let $(G, <)$ be an ordered nilpotent group of class 2 and let S be a subset of G of size $k \geq 4$, with $\langle S \rangle$ non-abelian. Write*

$S = \{x_1, x_2, \dots, x_{k-1}, x_k\}$ and $T = \{x_1, x_2, \dots, x_{k-1}\}$. Suppose that $\langle T \rangle$ is abelian and

$$|S^2| = 3k + i \leq 4k - 6$$

for some integer i satisfying $i > -k$. Then

$$S \subseteq \{a, ac, \dots, ac^{k+i}, b\} \subseteq \{a, ac, \dots, ac^{2k-6}, b\}$$

where $a, b, c \in G$, $c > 1$ and either $ab = bac^v$ or $ba = abc^v$ for some $v \in \mathbb{N}$ satisfying $v \leq k + i$.

Proof. Since $\langle T \rangle$ is abelian and $\langle S \rangle$ is non-abelian, it follows that $x_k \notin C_G(T)$ and hence $|x_k T \cup T x_k| \geq k$ by Proposition 2.4 of [7]. Moreover,

$$S^2 = (T^2 \dot{\cup} \{x_k^2\}) \dot{\cup} (x_k T \cup T x_k)$$

since $x_k \notin \langle T \rangle \subseteq C_G(T)$, and $x_k^2 \notin T^2$ since otherwise $x_k^2 \in C_G(T)$, which implies that $x_k \in C_G(T)$, a contradiction. Thus

$$3k + i = |S^2| = |T^2| + 1 + |x_k T \cup T x_k| \geq |T^2| + 1 + k, \quad (1)$$

and consequently

$$|T^2| \leq 2k + i - 1 \leq 3k - 7 = 3|T| - 4.$$

Hence it follows by Proposition 3.1 in [7] that

$$T \subseteq \{a, ac, \dots, ac^{k+i}\} \subseteq \{a, ac, ac^2, \dots, ac^{2k-6}\},$$

where $a, c \in G$, $c > 1$ and $ac = ca$. Moreover, as $|T^2| \geq 2|T| - 1 = 2(k-1) - 1$, our assumptions and (1) also imply that

$$4k - 6 \geq 3k + i = |S^2| \geq |T^2| + 1 + |x_k T \cup T x_k| \geq (2(k-1) - 1) + 1 + |x_k T \cup T x_k|,$$

so

$$2|T| - |x_k T \cap T x_k| = |x_k T \cup T x_k| \leq 2k - 4$$

and

$$|x_k T \cap T x_k| \geq 2(k-1) - (2k-4) = 2. \quad (2)$$

Write $x_k = b$. Then

$$bT \subseteq \{ba, bac, \dots, bac^{k+i}\} \text{ and } Tb \subseteq \{ab, cab, c^2ab, \dots, c^{k+i}ab\},$$

in view of $ac = ca$. As, by (2), $|bT \cap Tb| \geq 2$, there exist $0 \leq l, j, s, t \leq k + i$ such that

$$bac^l = c^j ab \quad \text{and} \quad bac^s = c^t ab,$$

with $l \neq s$ and $j \neq t$. Now, $bac^l = ac^j b$ implies that $b^{-1}a^{-1}bac^l = b^{-1}c^j b$, yielding

$$[b, a] = b^{-1}c^j b c^{-j} c^{j-l} = [b, c^{-j}] c^{j-l}.$$

Hence $c^{j-l} \in Z(G)$ and similarly $c^{s-t} \in Z(G)$.

Suppose, first, that $j \neq l$. Then $c^{j-l} \in Z(G)$ implies that $c \in Z(G)$. If $l > j$, then

$$ab = bac^{l-j} \quad \text{with} \quad 0 < l - j \leq k + i,$$

and if $j > l$, then

$$ba = abc^{j-l} \quad \text{with} \quad 0 < j - l \leq k + i.$$

Thus the theorem holds. Similarly, the theorem holds if $l > j$ and if $s \neq t$.

So assume, finally, that $l = j$ and $s = t$. In this case we shall reach a contradiction. We have $bac^l = ac^l b$, $bac^s = ac^s b$ and $l \neq s$. Thus

$$1 = [b, ac^l] = [b, a][b, c^l] \quad \text{and} \quad 1 = [b, ac^s] = [b, a][b, c^s].$$

Hence $[b, c^l] = [b, c^s]$, implying that $c^{-l}bc^l = c^{-s}bc^s$. Thus $c^{l-s} \in C_G(b)$ and since $l \neq s$, it follows that $c \in C_G(b)$ and $b \in C_G(c)$. But then $bac^l = abc^l$ and $b \in C_G(a)$. So $b \in C_G(T)$, a contradiction. \square

Notice that, conversely, if $S = \{a, ac, ac^2, \dots, c^{k-2}, b\}$, with $k \geq 3$, $c > 1$ and either $ab = bac^v$ or $ba = abc^v$ for some $v \in \mathbb{N}$ satisfying $0 < v \leq k - 3$, then, by Lemma 2.3, $|S^2| = 3k + v - 3 \leq 4k - 6$.

3. Subsets S with $|S^2| = 3|S| - 2$.

The aim of this section is to describe subsets S of a torsion-free nilpotent group of class 2 satisfying $|S| = k$ and $|S^2| = 3k - 2$.

If $|S| = 2$ and $|S^2| = 3 \cdot 2 - 2 = 4$, then S is completely non-abelian, and the converse is also true by Proposition 2.2. So we shall study subsets S with $|S| = k \geq 3$.

Using Lemma 2.4, it is easy to construct subsets S of a torsion-free nilpotent group of class 2 such that $|S| = k$ and $|S^2| = 3k - 2$. In fact, we have:

Example 3.1 *Let G be a torsion-free nilpotent group of class 2 and let $a, b, c \in G$ with $c > 1$ and either $ab = bac$ or $ba = cab$.*

Consider the subset

$$S = \{a, ac, ac^2, \dots, ac^i, b, bc, bc^2, \dots, bc^j\},$$

with i, j non-negative integers and $1 + i + 1 + j = k \geq 3$. Then $|S| = k$ and, by Lemma 2.4, $|S^2| = 3k - 2$.

Our main result in this paper is the following theorem.

Theorem 3.2 *Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $k \geq 4$ with $\langle S \rangle$ non-abelian. Then $|S^2| = 3k - 2$ if and only if*

$$S = \{a, \dots, ac^i, b, \dots, bc^j\},$$

where $a, b, c \in G$, i, j are non-negative integers satisfying $1 + i + 1 + j = k$, $c > 1$ and either $ab = bac$ or $ba = abc$.

In the case when $k = 3$, we have the following result.

Proposition 3.3 *Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $|S| = 3$, with $\langle S \rangle$ non-abelian. Then $|S^2| = 7$ if and only if one of the following holds:*

- (i) $S \cap Z(\langle S \rangle) \neq \emptyset$;
- (ii) $S = \{a, ac, b\}$, where $a, b, c \in G$, $c > 1$ and either $ab = bac$ or $ba = abc$. In particular, $c \in Z(G)$.

First we prove Proposition 3.3.

Proof. There exists an order $<$ on G such that $(G, <)$ is an ordered group. Write $S = \{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$ and $T = \{x_1, x_2\}$.

Suppose that $|S^2| = 7$. If $x_1x_2 = x_2x_1$ and $x_2x_3 = x_3x_2$, then $x_2 \in Z(\langle S \rangle)$ and S satisfies (i).

So assume, first, that $x_2x_3 \neq x_3x_2$. Since $|T^2| \leq |S^2| - 4 = 3$ by Lemma 2.1, it follows that $x_1x_2 = x_2x_1$. If $x_1x_3 = x_3x_1$, then $x_1 \in Z(\langle S \rangle)$ and S satisfies (i). So assume that $x_1x_3 \neq x_3x_1$. Since $x_2x_3 \neq x_3x_2$, it follows that $x_3 \notin \langle x_1, x_2 \rangle$ and the elements

$$x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2$$

of S^2 are all different. Since $|S^2| = 7$, we have either $x_3x_1 = x_2x_3$ or $x_3x_2 = x_1x_3$. Thus, if we put $x_1 = a, x_2 = ac, x_3 = b$, then $c > 1$, $ac = ca$ and either $ba = cab$ or $bac = ab$. Hence $c \in Z(G)$ and S satisfies (ii).

Similarly, if $x_1x_2 \neq x_2x_1$, then we get the result by considering the order opposite to $<$.

A direct calculation yields the converse. □

In order to prove Theorem 3.2, we study first the following particular case.

Proposition 3.4 *Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $|S| = k \geq 4$, with $\langle S \rangle$ non-abelian.*

Assume that $|S^2| = 3k - 2$ and $S = T \cup \{x\}$, with $\langle T \rangle$ abelian. Then

$$S = \{a, ac, \dots, ac^{k-2}, b\},$$

where $c > 1$ and either $ba = abc$ or $ab = bac$. In particular, $c \in Z(G)$.

Proof. By Theorem 2.5, we have

$$S \subseteq \{a, ac, \dots, ac^{k-2}, b\},$$

with $c > 1$ and either $ab = bac^v$ or $ba = abc^v$ for some $v \in \mathbb{N}$ satisfying $0 < v \leq k - 2$. Since $|S| = k$, it follows that $S = \{a, ac, \dots, ac^{k-2}, b\}$, and by Lemma 2.3 we get $|S^2| = 3k + (v - 3)$. But $|S^2| = 3k - 2$, so $v = 1$, as required. \square

Now we can prove Theorem 3.2.

Proof. Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of size $k \geq 4$ with $\langle S \rangle$ non-abelian.

If $a, b, c \in G$ with $c > 1$ and either $ab = bac$ or $ba = cab$, and if $S = \{a, ac, \dots, ac^i, b, bc, bc^2, \dots, bc^j\}$ with i, j denoting non-negative integers satisfying $1 + i + 1 + j = k \geq 4$, then $|S^2| = 3k - 2$ by Example 3.1.

Conversely, assume that $|S^2| = 3k - 2$. Our aim is to prove that there exist $a, b, c \in G$ such that $S = \{a, ac, \dots, ac^i, b, bc, bc^2, \dots, bc^j\}$, where $c > 1$ and either $ab = bac$ or $ba = abc$, and where i, j are non-negative integers satisfying $1 + i + 1 + j = k \geq 4$.

There exists an order $<$ on G , such that $(G, <)$ is an ordered group. Write

$$S = \{x_1, x_2, \dots, x_k\}, \quad T = \{x_1, \dots, x_{k-1}\}, \quad V = \{x_2, \dots, x_k\},$$

and suppose that $x_1 < x_2 < \dots < x_k$. If S contains a subset of size $k - 1$ which generates an abelian subgroup of G , then our claim follows by Proposition 3.4. Therefore we may assume that S contains no subsets of size $k - 1$ generating an abelian subgroup of G . In particular, the subgroups $\langle T \rangle$ and $\langle V \rangle$ of G are non-abelian.

If $x_{k-1}x_k \neq x_kx_{k-1}$, then $\langle T \rangle$ is abelian by Lemma 2.1, a contradiction. So we may assume that

$$x_{k-1}x_k = x_kx_{k-1}.$$

Similarly, by considering the order opposite to $<$ and the set V , we may assume that

$$x_1x_2 = x_2x_1.$$

Obviously $\{x_k^2, x_{k-1}x_k, x_kx_{k-1}\} \cap T^2 = \emptyset$. Let $\mu + 1$ be a minimal integer such that $\langle x_{\mu+1}, \dots, x_k \rangle$ is abelian. By our assumptions, $0 < \mu \leq k - 2$.

If $x_\mu x_k$ and $x_k x_\mu$ both belong to T^2 , then $x_\mu x_k = x_l x_m$, with $\mu < l \leq k - 1$ and $m < k$, and $x_k x_\mu = x_s x_t$, with $s < k$ and $\mu < t \leq k - 1$. If either $m > \mu$ or $s > \mu$, then $x_\mu \in \langle x_{\mu+1}, \dots, x_k \rangle$ and $\langle x_\mu, \dots, x_k \rangle$ is abelian, in contradiction to the minimality of $\mu + 1$. Therefore $m \leq \mu$ and $s \leq \mu$, yielding $x_\mu x_k = x_l x_m \leq x_{k-1} x_\mu$ and $x_k x_\mu = x_s x_t \leq x_\mu x_{k-1}$. It follows from $x_\mu x_k \leq x_{k-1} x_\mu = x_\mu x_{k-1} [x_{k-1}, x_\mu]$ that $[x_{k-1}, x_\mu] > 1$. Thus $[x_\mu, x_{k-1}] < 1$ and

$$x_k x_\mu \leq x_\mu x_{k-1} = x_{k-1} x_\mu [x_\mu, x_{k-1}] < x_{k-1} x_\mu,$$

a contradiction. Hence either $x_\mu x_k \notin T^2$, or $x_k x_\mu \notin T^2$. So either $S^2 \supseteq T^2 \cup \{x_k^2, x_{k-1}x_k, x_\mu x_k\}$ or $S^2 \supseteq T^2 \cup \{x_k^2, x_kx_{k-1}, x_kx_\mu\}$, which implies that $3k - 2 = |S^2| \geq |T^2| + 3$. Thus

$$|T^2| \leq 3k - 5 = 3(k - 1) - 2 \text{ and similarly } |V^2| \leq 3(k - 1) - 2.$$

Since if $|T^2| \leq 3(k - 1) - 3$, then $\langle T \rangle$ is abelian by Theorem 1.3 of [7] in contradiction to our assumptions, we may conclude that $|T^2| = 3(k - 1) - 2$. Similarly, also $|V^2| = 3(k - 1) - 2$.

Moreover, we may assume that

$$|\{x_1x_k, x_2x_k, \dots, x_{k-2}x_k\} \setminus T^2| < 2, \quad (3)$$

since otherwise, in view of $x_{k-1}x_k, x_k^2 \notin T^2$, we obtain $3k - 2 = |S^2| \geq |T^2| + 4$, yielding $|T^2| \leq 3k - 2 - 4 = 3(k - 1) - 3$. But then, again by Theorem 1.3 of [7], $\langle T \rangle$ is abelian, in contradiction to our assumptions. A similar argument indicates that

$$|\{x_kx_1, x_kx_2, \dots, x_kx_{k-2}\} \setminus T^2| < 2. \quad (4)$$

We now argue by induction on k .

If $k = 4$, then $|T| = 3$, $|T^2| = 7$ and we may apply Proposition 3.3.

We will first show that $T \cap Z(\langle T \rangle) = \emptyset$.

Assume that $T \cap Z(\langle T \rangle) \neq \emptyset$. Recall that we have $x_1x_2 = x_2x_1$, $x_3x_4 = x_4x_3$ and by (3) either $x_1x_4 \in T^2$ or $x_2x_4 \in T^2$. In any case $x_4 \in \langle T \rangle$.

Now, if $x_3 \in Z(\langle T \rangle)$, then $\langle x_1, x_2, x_3 \rangle$ is abelian, in contradiction to our assumptions. If $x_r \in Z(\langle T \rangle)$ for $r \in \{1, 2\}$, then $x_rx_3 = x_3x_r$ and $x_rx_4 = x_4x_r$, since $x_4 \in \langle T \rangle$, so $\langle x_r, x_3, x_4 \rangle$ is abelian, again in contradiction to our assumptions. Thus $T \cap Z(\langle T \rangle) \neq \emptyset$ is impossible.

Next assume that $T = \{a, ac, b\}$, with $c > 1$ and either $ab = bac$ or $ba = abc$. In particular, $c \in Z(G)$. We have $x_3x_4 = x_4x_3$. If either $x_3 = a$ or $x_3 = ac$, then $\langle a, ac, x_4 \rangle$ is abelian, in contradiction to our assumptions. So assume that $x_3 = b$. We have $bx_4 = x_4b$ and

by (3) either $ax_4 \in T^2$ or $acx_4 \in T^2$. If either ax_4 or acx_4 belongs to $\{a^2, a^2c, a^2c^2, b^2\}$, then $bx_4 = x_4b$ implies that $ab = ba$, a contradiction. Since $x_4 \neq b$, it follows that either $ax_4 \in \{abc, ba, bac\}$ or $acx_4 \in \{ab, ba, bac\}$. But either $ba = abc$ or $ba = abc^{-1}$, so $x_4 \in \{bc, bc^2, bc^{-1}, bc^{-2}\}$. Since $c^{-1} < 1$ and $x_4 > b$, $x_4 = bc^{-1}$ and $x_4 = bc^{-2}$ are impossible. So $x_4 \in \{bc, bc^2\}$.

If $x_4 = bc^2$, then $S = \{a, ac, b, bc^2\}$ and

$$S^2 = \{a^2, a^2c, ab, abc^2, a^2c^2, abc, abc^3, b^2, b^2c^2, abc^4, b^2c^4\}$$

if $ba = abc$, and

$$S^2 = \{a^2, a^2c, ab, abc^2, a^2c^2, abc, abc^3, abc^{-1}, b^2, b^2c^2, b^2c^4\}$$

if $ba = abc^{-1}$. It is easy to see that the elements in each of the above two sets are distinct from each other, yielding $|S^2| = 11 = 3|S| - 1$, a contradiction. Hence $x_4 = bc$ and our claim follows.

Now assume that $k > 4$ and, by induction, the result is true for $k-1$. Then there exist $a, b, c \in G$ such that

$$T = \{a, ac, \dots, ac^i, b, bc, \dots, bc^j\},$$

where $c > 1$ and either $ab = bac$ or $ba = abc$, and where i, j are non-negative integers satisfying $1 + i + 1 + j = k - 1 \geq 4$. In particular, $c \in Z(G)$ and since x_{k-1} is a maximal element of T , we have either $x_{k-1} = ac^i$, $i \geq 0$ or $x_{k-1} = bc^j$, $j \geq 0$. Assume, without loss of generality, that $x_{k-1} = bc^j$. Then

$$[b, x_k] = 1,$$

since $x_k x_{k-1} = x_{k-1} x_k$. We may also assume that

$$i \geq 1,$$

since otherwise $S = \{a, b, bc, \dots, bc^j, x_k\}$ with $\langle b, bc, \dots, bc^j, x_k \rangle$ abelian, in contradiction to our assumptions. Thus $ac^{i-1}, ac^i \leq x_{k-2}$ and by (3) either $ac^i x_k \in T^2$ or $ac^{i-1} x_k \in T^2$. Therefore we have either $ac^i x_k \in \{a^2 c^r, abc^s, b^2 c^v, bac^s\}$ or $ac^{i-1} x_k \in \{a^2 c^r, abc^s, b^2 c^v, bac^s\}$, where r, s, v are non-negative integers satisfying $r \leq 2$, $s \leq i + j$ and $v \leq 2j$.

If either $ac^i x_k = a^2 c^r$ or $ac^i x_k = b^2 c^v$, then either $x_k = ac^{r-i}$ and $[a, b] = [x_k, b] = 1$, a contradiction, or $ax_k = b^2 c^{v-i}$ and $1 = [ax_k, b] = [a, b]$, again a contradiction. Similarly, also $ac^{i-1} x_k \in \{a^2 c^r, b^2 c^v\}$ is impossible. Therefore one of the following four equalities holds:

$$ac^i x_k = abc^s, \quad ac^i x_k = bac^s, \quad ac^{i-1} x_k = abc^s, \quad ac^{i-1} x_k = bac^s$$

with $0 \leq s \leq i + j$. Consequently, if $ba = abc$, then one of the following three equalities holds:

$$x_k = bc^{s-i}, \quad x_k = bc^{s-i+1}, \quad x_k = bc^{s-i+2}$$

and if $ba = abc^{-1}$, then one of the following three equalities holds:

$$x_k = bc^{s-i}, \quad x_k = bc^{s-i-1}, \quad x_k = bc^{s-i+1}.$$

Thus

$$x_k = bc^l, \text{ with } l \leq j+2 \text{ if } ba = abc \text{ and } l \leq j+1 \text{ if } ab = bac.$$

But we also know, by (4), that either $x_k ac^i \in T^2$ or $x_k ac^{i-1} \in T^2$. Thus either $x_k ac^i \in \{a^2 c^r, abc^s, b^2 c^v, bac^s\}$ or $x_k ac^{i-1} \in \{a^2 c^r, abc^s, b^2 c^v, bac^s\}$, where r, s, v are non-negative integers satisfying $r \leq 2i$, $s \leq i+j$ and $v \leq 2j$. Arguing as before, it follows that one of the following four equalities holds:

$$x_k ac^i = abc^s, \quad x_k ac^i = bac^s, \quad x_k ac^{i-1} = abc^s, \quad x_k ac^{i-1} = bac^s$$

with $0 \leq s \leq i+j$. Consequently, if $ab = bac^{-1}$, then one of the following three equalities holds:

$$x_k = bc^{s-i-1}, \quad x_k = bc^{s-1}, \quad x_k = bc^{s-i+1}$$

and if $ab = bac$, then one of the following three equalities holds:

$$x_k = bc^{s-i}, \quad x_k = bc^{s-i+1}, \quad x_k = bc^{s-i+2}.$$

Thus

$$x_k = bc^l, \text{ with } l \leq j+1 \text{ if } ba = abc \text{ and } l \leq j+2 \text{ if } ab = bac.$$

It follows that

$$x_k = bc^l \quad \text{with} \quad l \leq j+1.$$

Since $c > 1$ and $b < x_k$, we must have $l > 0$ and $x_k \notin T$ implies that $l = j+1$. Hence $x_k = bc^{j+1}$. This proves our claim and completes the proof of the theorem. □

ACKNOWLEDGEMENTS

This work was supported by the “National Group for Algebraic and Geometric Structures, and their Applications” (GNSAGA - INDAM), Italy.

The second author is grateful to the Department of Mathematics of the University of Salerno for its hospitality and support, while this investigation was carried out.

REFERENCES

- [1] L.V. Brailovsky, G.A. Freiman, On a product of finite subsets in a torsion-free group *J. Algebra* **130** (1990), 462–476.
- [2] E. Breuillard, B.J. Green and T.C. Tao, The structure of approximate groups, *Publ. Math. IHES.* **116** (2012), 115–221.
- [3] G.A. Freiman, On the addition of finite sets. I., *Izv. Vyss. Ucebn. Zaved. Matematika* **6** (13) (1959), 202–213.
- [4] G.A. Freiman, Groups and the inverse problems of additive number theory, Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), Kalinin. Gos. Univ., Moscow, (1973)175183 (Russian)
- [5] G.A. Freiman, Structure Theory of Set Addition, *Astérisque* **258** (1999),1–33.
- [6] G.A. Freiman, On finite subsets of nonabelian groups with small doubling, *Proc. of the Amer. Math. Soc.* **140**, no. 9 (2012), 2997–3002.
- [7] G. Freiman, M. Herzog, P. Longobardi, M. Maj, Small doubling in ordered groups, *J. Aust. Math. Soc.* **96**, no. 3 (2014), 316–325.
- [8] G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Direct and inverse problems in additive number theory and in non abelian group theory, *European J. Combin.* **40** (2014), 42–54.
- [9] G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, A small doubling structure theorem in a Baumslag-Solitar group, *European J. Combin.* **44** (2015), 106–124.
- [10] G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, On the structure of subsets of an orderable group with some small doubling properties, *J. Algebra* **445** (2016), 307–326.
- [11] B. Green, What is ... an approximate group?, *Notices Amer. Math. Soc.* **59** (2012), no. 5, 655–656.
- [12] B. Green, I. Z. Ruzsa, Freiman’s theorem in an arbitrary abelian group, *J. London Math. Soc.* **75** (2007), no. 1, 163–175.
- [13] Y.O. Hamidoune, A.S. Lladó, O. Serra, On subsets with small product in torsion-free groups, *Combinatorica* **18** (1998), 529–540.
- [14] A.I. Mal’cev, On ordered groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **13** (1948), 473–482.
- [15] M.B. Nathanson, Additive Number Theory Inverse Problems and the Geometry of Sumsets, Springer, (1996).
- [16] B.H. Neumann, On ordered groups, *Amer. J. Math.* **71** (1949), 1–18.
- [17] I.Z. Ruzsa, An analog of Freiman’s theorem in groups, *Astérisque* **258** (1999), 323–326.
- [18] T. Sanders, The structure theory of set addition revisited, *Bull. Amer. Math. Soc. (N.S.)* **50** (2013), no. 1, 93–127.
- [19] Y.V. Stanchescu, The structure of d -dimensional sets with small sumset, *J. Number Theory* **130** (2010), no. 2, 289–303.
- [20] T.C. Tao, Product set estimates for noncommutative groups, *Combinatorica* **28** (2008), no. 5, 547–594.

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